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PLASTIC BUCKLING OF ECCENTRICALLY LOADED ALUMINUM ALLOY COLUMNS

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PLASTIC BUCKLING OF ECCENTRICALLY LOADED ALUMINUM ALLOY COLUMNS

J. W. Clark[1], J. M. ASCE

I. Introduction

If a member subjected to combined compressive and load and bending is so proportioned that it does not fail by lateral torsional buckling or because of local buckling, failure will result from instability caused by plastic yielding of the material. This report describes an experimental investigation of the latter type of failure, in which the combined loading was obtained by testing specimens as eccentrically loaded columns with simply supported ends.

The principal object of this investigation was to obtain critical values of combined end load and bending moment that could be used to check design formulas for this type of member. An additional objective was to obtain records of the behavior of the test specimens as they reached and passed the critical load. For this purpose, continuous measurements of load, deflection, and strain were recorded on a multi-channel oscillograph. The results of these measurements were compared with the behavior calculated on the basis of the stress-strain characteristics of the material.

II. Description of Specimens

The column specimens were made from aluminum alloy 61S-T6 rolled rectangular bar and drawn rectangular tube. This alloy is one that is commonly used for structural applications. The bar was selected to represent the case of a solid section and the tube was chosen as a section whose resistance to bending was provided primarily by the material in the flanges. Average cross-sectional dimensions are shown in Fig. 1. Dimensions of individual specimens were all within 1 % of the values listed. The areas of the rectangular bar specimens were 2.55 sq. in. for specimens B1, B2, and B5, and 2.64 sq. in. for the others. The least radius of gyration for all bar specimens was 0.432 in. The rectangular tube specimens were 2.17 sq. in., in area, with a least radius of gyration of 0.757.

Average mechanical properties in the longitudinal direction for the two types of specimens are listed in Table I. These values are typical of those commonly found in commercial products of this alloy. Values of tensile and compressive yield strengths for individual specimens were within 1% of the average values, and the variation of the tensile strength from the averages given in the table did not exceed 1.2 per cent. Compressive stress-strain curves for the two types of specimens are shown in Fig. 2.

The various specimen lengths and eccentricities of loading are listed in Table II. Values of slenderness ratio (length divided by radius of gyration) ranged from about 35 to 110 for the rectangular bar specimens and from 26

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to 89 for the rectangular tubes. The eccentricities, which were determined from the measured strains and deflections, as discussed in Appendix A, varied from zero to slightly less than one-half the depth of the cross section. The direction of the eccentricity was normal to the axis of least stiffness in all cases; that is, the specimens were forced to bend about the vertical axes shown in Fig. 1.

Dial gage* measurements of crookedness, with the specimens supported as beams above a flat surface, indicated that the crookedness, considered as the deviation of the center from a straight line through the ends, was less than 1/1000 of the length for all specimens.

III. Description of Tests

The columns were tested in a 300,000 lb capacity Amsler testing machine. Loads were increased continuously to failure in all cases. Initial loads were one kip for some specimens and five kips for others.

Three types of end support were employed, as indicated by the footnotes in Table II. The shorter specimens were tested on hydraulically supported, spherically seated platens;[1] shown in the photograph in Fig. 3. The advantages of these loading fixtures are that they combine a large loading capacity with low resistance to rotation and the centers of rotation are in the faces of the platens. The spherically seated platens were not used with the longer specimens because these platens cannot tip more than five degrees. In order to accommodate the end rotations of the longer specimens, it was necessary to use the knife-edge platens, shown in Fig. 4, which permitted 15° of rotation. The centers of rotation of the knife-edge platens were 1.25" from the faces of the platens. This factor was taken into account in the analysis of the test data by assuming the length of these specimens to be the distance between the centers of rotation of the knife-edge platens.[2]

After the original series of tests, two of the specimens, one bar and one tube, were retested with knife-edges machined on the ends of the specimens at a distance of 0.15 in. from one edge. The knife-edges were rounded to a radius of 1/16 in. and were supported in 120° grooves in bearing plates. The purpose of this method of loading was to obtain large deflections without the danger of the specimens slipping off the platens. Although these specimens had received some slight permanent set in their initial loading, their crookedness still did not exceed 1/1000 of the length.

Two methods were employed in measuring the lateral deflection of the specimens. For the shorter members, deflection was measured by means of slender beams supported against the columns of the testing machine as shown in Fig. 3. These beams were attached to the column specimens by rods and yokes, so that the beams were forced to bend as the columns deflected. Type AB-1 SR-4 electrical-resistance wire strain gages cemented to the beams indicated the resulting bending strain, which was calibrated in terms of lateral deflection of the column specimens by recording the strain signals for various known values of deflection as measured by a dial gage.**

* A Federal Model D81S gage graduated to 0.001 in. was used.

1. "Hydraulically Supported Spherically Seated Compression Testing Machine Platens" by R. L. Templin, Proceedings, ASTM, Vol. 42, 1942.
2. The error involved in this assumption is negligible. See "Corrections for Lengths of Columns Tested Between Knife-Edges" by W. R. Osgood, Journal of the Aeronautical Sciences, Vol. II, No. 4, Oct. 1944, p. 378.

** A Federal Model D81S gage graduated to 0.001 in. was used.

The beams described above were designed to be very flexible in order to minimize the forces they exerted on the column specimens. Calculations showed that the bending moments induced in the specimens by the lateral forces required to deflect the beams did not exceed 0.01 per cent of the bending moments produced by the end loads on the specimens.

In the tests of the longer columns, continuous rotating potentiometers with pulleys were used to indicate deflections. The pulleys were actuated by 0.005 in. diameter wires attached to yokes at the center and one quarter point of the column specimens. The electrical signal from these potentiometers was calibrated against a deflection measured with a 0.001 in. micrometer screw.

Measurement of deflection as described above requires that the testing machine platens do not move relative to the columns of the machine. This movement was checked by dial gage* measurements and found to be negligible for loads exceeding 10 kips. At lower loads there was some shifting of the platens, and for this reason the "zero" load for the deflection measurements was taken to be 10 kips.

Strains were measured on all four sides of the specimens at the midlength and on two sides at one quarter point. The gages employed were type A-11 SR-4 electrical-resistance wire strain gages.

To indicate load, a Baldwin SR-4 Fluid Pressure Cell of 5,000 psi capacity was connected into the hydraulic circuit of the testing machine. Values of maximum load were read from the load-indicating dial of the testing machine, which periodic calibrations have shown to be accurate within ±1.0 per cent. These maximum load values were used to calibrate the signals from the pressure cell. The signal from this cell, as well as the strain and deflection signals, were recorded on a Hathaway type S8-C recording oscillograph.

Before loading to failure, the specimens for which the eccentricities were less than 0.010 in. were subjected to loadings in the elastic stress range while strains were recorded statically. With the aid of these measurements the specimens were located on the platens so as to minimize the effects of crookedness and unintentional eccentricities.

IV. Results and Discussion

The maximum loads attained in the various tests are listed in Table II. Figs. 5, 6, and 7 show some of the strain and deflection data for a typical specimen, No. B12. While it is not the intention in this paper to discuss all the data recorded in these tests, Figs. 5, 6, and 7 will serve to illustrate the behavior of the specimens and the type of information obtained.

1. Strains and Deflections

The observed strains and deflections for specimen B12 are compared in Figs. 5, 6, and 7 with various calculated values. The calculated curves listed as "elastic" in Fig. 5, 6, and 7 are based on the equation:[3]

$$y_o = e\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_E}} - 1\right) \qquad (1)$$

*A Federal Model D81S gage, graduated to 0.001 in. was used.

3. This is the equation on which the secant column formula is based, see, for example, "Buckling Strength of Metal Structures" by F. Bleich, McGraw-Hill Book Co., Inc., New York, N. Y. 1952, p. 45.

where y_0 = deflection at center of column, in.
e = eccentricity, in.
P = load on column, kips
P_E = Euler critical load for column, kips

$$P_E = \pi^2 EI/L^2$$

E = Modulus of elasticity, kips per sq. in.
I = moment of inertia of column about axis normal to the plane of bending, in.^{4}
L = length of column, in.

In Fig. 8 is shown a diagramatic sketch of a column in a deflected position and also two mathematical expressions which may be assumed to represent the deflected shape of the column. The latter expressions were used in calculating the curves for plastic behavior plotted in Figs. 5, 6, and 7.

Methods for calculating relationships between load and deflection or strain for a column in the plastic stress range have been discussed by several authors.[4,5,6] The methods used here are discussed briefly in the following paragraphs and are presented in more detail in Appendix B.

From the stress-strain curve of the material, one can determine a relationship between the load, moment arm of the load (eccentricity plus deflection), and curvature for a given column cross section. The assumption of some particular shape of the column in the deflected position leads to a second relationship between curvature and moment arm. For a given load, one can find values of deflection that satisfy both the relationship based on the stress strain curve and the relationship based on the assumed shape of the column, and thus establish the calculated deflection of the column at the given load. The two sets of relationships are not satisfied at every point along the length of the column but only at a finite number of points, depending upon the number of arbitrary parameters in the mathematical expression assumed for the deflected shape of the column.

The cosine relationship listed as equation (2) in Fig. 8 is the same as that used by Westergaard and Osgood in an analysis of eccentrically loaded steel columns presented in 1928.[4] This curve is, of course, the correct one when the stresses are in the elastic stress range. For stresses in the plastic range, this curve provides a relatively simple solution involving only one arbitrary parameter. As can be seen in Figs. 5 and 6, the strains and deflections at the center calculated on the basis of this assumption are not greatly in error. The maximum load indicated by the calculated curves is about 2.7 per cent less than the measured value.

Better agreement between measured and calculated values can be achieved by assuming a more complicated deflection curve, which can be made to fit more closely the actual shape of the deflected member. The expression listed as equation (3) in Fig. 8, which incorporates three arbitrary parameters, yielded very close agreement between measured and calculated strains and deflections.

A rather sensitive indication of the deflected shape of the column was obtained by comparing the curvatures at the quarter point and center as indicated

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4. "Strength of Steel Columns" by H. M. Westergaard and W. R. Osgood, Transactions, ASME, Vol. 50, 1928, APM-50-9, p. 65.
5. "Inelastic Column Behavior" by J. E. Duberg and T. W. Wilder, III, Technical Note No. 2267, National Advisory Committee for Aeronautics, Jan.,1951
6. "Buckling Strength of Metal Structures," by Friedrich Bleich, McGraw-Hill Book Co., Inc., New York, N. Y. 1952, p. 25.

by the strain measurements at those points. In Fig. 7 are plotted measured and computed values of the ratio of curvature at the quarter point to curvature at the center. Again the good agreement between measured values and those calculated on the basis of equation (3) in Fig. 8 can be observed.

2. Maximum Loads

Maximum loads for all specimens were calculated from the assumption that the column deflects in a cosine curve (equation (2), Fig. 8); These maximum loads are listed in Table II. The calculated critical loads averaged 2.4 per cent less than the measured values for the rectangular bar column specimens and 3.3 per cent less than the measured values for the tubes.

The above differences are of the same order of magnitude as the difference in maximum loads for the two sets of calculated curves in Figs. 5 and 6, so a part of the error in the calculated maximum loads in Table II was no doubt a result of the approximations involved in using the cosine curve (Equation (2), Fig. 8) to represent the deflected shape of the column in the plastic stress range.

An additional factor affecting the agreement between measured and calculated maximum loads is the end restraint provided by the loading fixtures. This restraint seems to have been appreciable only in the case of the specimens loaded with eccentricities less than 0.010 in. Comparison of critical loads for zero eccentricity, determined from the test data as described subsequently, with calculated critical loads at stresses in the elastic range indicated that the ratio between the effective length and the actual length of the specimens with the smallest eccentricities was about 0.985.* For the specimens with greater eccentricities, analysis of the measured strains and deflections showed that the ratio of effective length to actual length was even closer to unity, so that the effect of end restraint was negligible.

3. Secant Column Formula

One method of design for members of this type is based on the assumption that failure occurs when the extreme fiber stress reaches some limiting value. The maximum extreme fiber stress is assumed to be the sum of the axial stress plus the bending stress at the center, where the moment arm used in computing the bending stress is the eccentricity plus the center deflection as given by equation (1). The resulting expression for maximum stress is known as the secant formula[3].

On the basis of these test results, the above method does not appear to be satisfactory for application to aluminum alloy members. For example, if it is assumed that a member fails when the extreme fiber stress reaches the yield strength, the secant formula predicts critical loads for the rectangular tubes that vary from 5 per cent less than the experimental value for specimen T15, to 15 per cent greater than the actual ultimate strength for specimen T12. Critical loads calculated in the same way for the bars are from 7 per cent too high in the case of specimen B16 to 22 per cent too low for specimen B18.

4. Interaction Formulas

Another method of predicting the strength of beam-columns for design purposes involves the use of a so-called interaction formula. Such a formula expresses a relationship between two ratios: the ratio of axial load on the column to the load that would cause failure under axial compression alone, and the ratio of bending moment on the member to the moment that would

* This ratio is 1.0 for perfectly hinged ends and 0.5 for fixed ends.

cause failure under bending alone. To investigate these test results from the stand-point of interaction relationships, it was necessary to obtain values of critical axial load and critical bending moment. Values of critical axial load for each length of specimen were obtained by plotting test values of axial stress at failure versus eccentricity for the specimens of that length. The value of stress at which a curve drawn through the test points crossed the axis of zero eccentricity was taken to be the critical stress for a centrally loaded column of that length. The loads corresponding to these critical stresses were from zero to 2.9 per cent higher than the critical loads for the specimens with the smallest eccentricities.

An indication of the strength of these members in pure bending was obtained from the tests of specimens B4 and T2 with knife edges machined on the ends of the specimens. The results of these two tests are plotted in Fig. 9. In addition to curves of load versus center deflection, Fig. 9 shows the relationship between load and bending moment at the center for these specimens. Values of bending moment were obtained simply by multiplying the load times the sum of the initial eccentricity plus the measured deflection at that load. It can be seen from Fig. 9 that in the case of the rectangular tube the bending moment reached a definite maximum value. At this point the cross section of the tube at the center of the column was visibly distorted, both the tension and compression flanges being bent toward the neutral axis of the tube. The resulting reduction in stiffness of the section was sufficient to prevent any further increase in bending moment. Since at the end of this test the axial stress was relatively low, about 3.5 kips per sq. in., it was concluded that the value of maximum bending moment attained was influenced very little by the axial stress. This bending moment, 69.5 in. kips, was assumed to be the same as the critical value for pure bending.

The bending moment for specimen B4 did not reach a maximum value. This is, of course, typical of solid sections, which can be given very severe bends without causing failure. Therefore, it was necessary to pick a value of critical bending moment for the solid section on a somewhat arbitrary basis. The value of critical bending moment selected was one corresponding to a modulus of failure equal to 1.5 times the tensile strength of the material. This is a limiting value that is sometimes used in estimating ultimate strength for design purposes.[7] As indicated in Fig. 9, when a bending moment corresponding to a modulus of failure of 1.5 times the tensile strength was reached, the deflection of specimen B4 was about 4.2 in. or roughly one-tenth of the length of the specimen. The axial stress, about 3.5 kips per sq. in., was low enough that it was assumed to have relatively little effect on the bending strength.

The test results of this investigation are plotted in interaction form in Figs. 10 and 11. Also plotted in these figures are three different interaction curves that have been suggested as being applicable to the design of beam-columns. The algebraic expressions for these curves are listed beneath the diagrams in Fig. 10 and 11.

The quantities used in the interaction formulas may be defined as follows:

P = load on column at failure, kips

P' = maximum load for axial compression alone, kips

7. "Designing Aluminum Alloy Members for Combined End Load and Bending", by H. N. Hill, E. C. Hartmann, and J. W. Clark, ASCE Proceedings, Separate No. 300.

M = bending moment at failure (for these specimens end load times initial eccentricity), in.-kips

M' = maximum bending moment for bending alone, in.-kips

A = area of cross section, sq. in.

S = section modulus of cross section, in.^{3}

P_E = Euler critical load for column, kips

Formula (4) in Figs. 10 and 11 has served as a basis for at least one widely-used design formula for steel members subjected to combined loading.[8] This formula gives a straight diagonal line on the interaction diagram. The fact that all the test points lie below this line in Fig. 10 and 11 indicates that formula (4) is unconservative. Critical loads predicted by this formula were as much as 39 per cent higher than the test values.

Formula (5) in Figs. 10 and 11 is similar to a design formula that has been used in specifications for aluminum alloy structures.[9,10] Although this formula is an improvement over the straight line relationship, it predicts critical loads that are from 10 per cent lower to 31 per cent higher than the measured values for the specimens of this investigation.

The last formula listed in Figs. 10 and 11, Formula (6), was originally developed for application to aluminum alloy beam-columns that fail by lateral buckling and twisting.[11] It has also been suggested as a possible basis for the design of steel beam-columns.[12] For the column tests described herein, values of ultimate strength predicted by formula (6) were within $\pm$ 10 per cent of the measured maximum loads. This agreement is considerably better than that obtained with either of the other two interaction formulas or the secant formula.

In view of this agreement, Formula (6) is recommended for use in the design of eccentrically loaded aluminum alloy columns that fail by plastic buckling in the direction of the applied bending moment, as well as for members that fail by lateral buckling and twisting.

Although these tests were confined to cases in which the bending moment was applied by eccentric loading with equal eccentricities at the two ends, it is believed that the recommended design formula should be satisfactory for any case of a simply supported aluminum alloy column of uniform section subjected to axial load and bending, when the maximum applied bending

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8. "Steel Construction," Manual of the American Institute of Steel Construction, 5th edition, 1950, page 284.
9. "Specifications for Heavy Duty Structures of High-Strength Aluminum Alloy, "Final Report of the Committee of the Structural Division on Design in Lightweight Structural Alloys, Transactions, ASCE, Vol. 117, 1952, p. 1258, paragraph B-7a.
10. "Specifications for Structures of a Moderate Strength Aluminum Alloy of High Resistance to Corrosion," Progress Report of the Committee of the Structural Division on Design in Lightweight Structural Alloys, ASCE Proceedings, Separate No. 132, 1952, p. 6, paragraph B-7a.
11. "Lateral Buckling of Eccentrically Loaded I- and H-Section Columns", by H. N. Hill and J. W. Clark, Proceedings of the First National Congress of Applied Mechanics, 1952, p. 407.
12. "Investigation of Interaction Formula," Report No. 1 to Column Research Council, by J. Zickel and D. C. Drucker, Brown University, April, 1951.

moment occurs at or near the center of the member and failure is of the type considered in this paper. This includes not only the column with equal eccentricities, but also columns in which bending is introduced by initial crookedness, by uniformly distributed lateral load, or by concentrated lateral loads distributed about the center in a fairly symmetrical manner.

Values of combined end load and bending moment that will cause failure of a member are somewhat smaller when the applied bending moment is constant along the length (equal eccentricities), rather than varying from zero at the ends to a maximum near the center as is the case when the moment is introduced by lateral loading or initial crookedness. Differences in strength corresponding to the various types of loading are relatively small, however, because the relationship between deflection of a member and curvature at the center, which largely governs the maximum load for a beam-column, is not greatly affected by variations in the shape of the bending moment diagram. Thus, a design formula that is applicable to eccentrically loaded columns would be expected to give reasonably accurate results, somewhat on the safe side, for the other types of loading.

Evidence in support of the above conclusions appears in the work of Westergaard and Osgood,[4] who found that a column subjected to a given combination of axial load and bending moment at the center has about the same ultimate strength regardless of whether the applied bending moment is introduced by equal eccentricities at the ends or by initial curvature.

V. Conclusions

The results of this investigation of beam-column behavior, in which tests were made on eccentrically loaded columns made from aluminum alloy 61S-T6 rolled rectangular bar and drawn rectangular tube, may be summarized as follows:

1. Values of slenderness ratio ranged from about 35 to 110 for the rectangular bar specimens and from 26 to 89 for the rectangular tubes. Eccentricities varied from zero to slightly less than one-half the depth of the cross section for each type of specimen.
2. All specimens failed by yielding, resulting in an eventual falling-off of the load accompanied by rapidly increasing deflections. The behavior is typified ty the curves of measured strain and deflection shown in Figs. 5, 6, and 7.
3. Measured strains and deflections in the plastic-stress range agreed approximately with values calculated on the basis of the assumption made by Westergaard and Osgood[3] that the deflected shape of the column is a part of a cosine curve (equation (2), Fig. 8). Maximum loads calculated on this basis averaged 2.4 per cent less than the measured critical loads for the rectangular-bar specimens and 3.3 per cent less than the test values for the rectangular tubes.
4. Agreement between measured and calculated behavior was improved by assuming the shape of the deflection curve to be represented by a parabola plus two terms of a sine series (equation (3), Fig. 8).
5. The assumption that failure occurs when the extreme fiber stress as calculated by the secant formula reaches the yield strength of the material resulted in calculated critical loads which were from 22% lower to 15% higher than the test values.
6. The straight-line interaction formula (Formula (4) in Figs. 10 and 11) predicted critical loads that were higher than the test values by as much as 39%.

7. Maximum loads calculated from interaction formula (5) in Figs. 10 and 11 varied from 10% lower to 31% higher than the measured values.

8. The interaction formula designated as formula (6) in Figs. 10 and 11 gave critical loads that agreed with the test results within plus or minus 10%. This formula is recommended as a basis for the design of eccentrically loaded aluminum alloy columns that fail by plastic buckling in the plane of the applied bending moment. The formula is believed to be applicable also to columns in which bending is introduced by initial crookedness or by lateral loads, as long as the maximum applied bending moment occurs at or near the center of the member.

APPENDIX A - DETERMINATION OF EFFECTIVE ECCENTRICITIES

The nominal eccentricities used in these tests were, for the bar specimens, zero, 0.05, 0.15, 0.25, 0.35, 0.45, and 0.60 in., and for the tubes, zero, 0.1, 0.3, 0.6, and 0.85 in. The last value given in each case is for the member tested with knife-edges machined on the ends. Tests of specimens of any given length did not necessarily include all eccentricities.

The actual effective eccentricities were somewhat modified by the effects of initial crookedness, and for this reason the values of eccentricity used in the analysis of the test data were calculated values based on the measured strains and deflections.

In determining the effective eccentricities, use was made of the following relationships based on equation (1):

$$e = \frac{y_o}{\sec \frac{\pi}{2}\sqrt{\frac{P}{P_E}} - 1} \qquad (7)$$

$$e = \frac{ES\,(\epsilon_1 - \epsilon_2)_o}{2P\,\sec \frac{\pi}{2}\sqrt{\frac{P}{P_E}}} \qquad (8)$$

where $(\epsilon_1 - \epsilon_2)_o$ = differnece in extreme fiber strains at the center of the column, in. per in.

S = section modulus of column, in.^{3}

The other quantities are as defined for equation (1).

If the numerator is plotted versus the demoninator in the expressions on the right-hand sides of the above equations, the resulting slope is the eccentricity. These equations apply, of course, only as long as the stresses are within the elastic range.

For the specimens with eccentricities less than 0.010 in., the values of eccentricity listed in Table II were calculated from equation (8), using the static strain measurements made while centering these specimens on the platens. These values of eccentricity are given to the nearest 0.001 in.

Most of the other values of eccentricity in Table II were computed on the basis of the strain data obtained from the oscillograph records. In a few cases where the strain data were incomplete (specimens B14, B9, and the two specimens with knife-edges machined on the ends), equation (7) was used with the measured deflections to determine the eccentricity. Eccentricities calculated from oscillograph record data are given to the nearest 0.005.in.

APPENDIX B - CALCULATION OF STRAINS AND DEFLECTIONS

The usual assumptions are made that deflections are small relative to the length of the member, shear deformations may be neglected, and plane sections remain plane. It is also assumed that the material follows the stress-strain relationships determined in tests under uniaxial tension or compression.

Given values of extreme fiber strain, ϵ_1 and ϵ_2, at some cross section of the column, one may determine the distribution of stress on the section from the stress-strain curve, and from this stress distribution calculate the axial load and the bending moment on the given cross section.*

Dividing the bending moment at the section by the load on the member gives a value of moment arm, which must be equal to the sum of the eccentricity plus the deflection at that cross section.

The given values of extreme fiber strain also establish the curvature at the section, the curvature being equal to $(\epsilon_1 - \epsilon_2)/d$, where d is the depth of the cross section. If the above calculations are repeated for a number of different assumed values of extreme fiber strains, the results may be plotted in dimensionless form as a family of curves relating the quantity $(\epsilon_1 - \epsilon_2)$ and the ratio of the moment arm to depth of section, $(e + y)/d$, for constant values of average stress on the column.

Another relationship between curvature and deflection is established by setting the second derivative of the assumed deflection curve equal to the curvature. The cosine curve, Equation(2) in Fig. 8, has only one arbitrary parameter and thus may be made to fit the relationship based on the stress-strain curve at only one point along the length of the column. It is convenient to choose this point as the center of the column, which gives the equation:

$$\frac{e + y_0}{d} = \frac{e}{d} \sec \sqrt{\frac{L^2 (\epsilon_1 - \epsilon_2)_0}{4d(e + y_0)}} \qquad (9)$$

The above equation can be plotted as a curve on the chart of $(\epsilon_1 - \epsilon_2)$ versus $(e + y)/d$. Points of intersection of this curve with the previously-drawn family of curves relating $(\epsilon_1 - \epsilon_2)$ and $(e + y)/d$ for constant values of average stress establish the deflections and extreme fiber strains corresponding to the various average stresses.

If equation (3) in Fig. 8 is used, the curvature-deflection relationship may be satisfied at three points along the length of the column. The end $(x = L/2)$, one-third point $(x = L/6)$, and center $(x = 0)$ were chosen, giving the following three equations:

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* The integration required to calculate the load and bending moment may be performed by numerical or graphical methods, or it may be carried out analytically by expressing the stress-strain curve in mathematical form. The latter method was used in this instance, the mathematical expression employed being the one developed by Ramberg and Osgood[13] and expressed in a modified form by H. N. Hill.[14]

13. "Description of Stress-Strain Curves by Three Parameters," by Walter Ramberg and W. R. Osgood, Technical Note 902, National Advisory Committee for Aeronautics, 1943.
14. "Determination of Stress-Strain Relations from Offset Yield Strength Values," by H. N. Hill, Technical Note 927, National Advisory Committee for Aeronautics, 1944.

$$\frac{e + y_{L/2}}{d} = \frac{e}{d} \tag{10a}$$

$$\frac{e + y_{L/6}}{d} = \frac{e}{d} + \frac{L}{(d)^2}\left[\left(\frac{1}{9} - \frac{1}{\pi^2}\right)(\epsilon_1 - \epsilon_2)_{L/2} + \frac{1}{\pi^2}(\epsilon_1 - \epsilon_2)_{L/6}\right] \tag{10b}$$

$$\frac{e + y_o}{d} = \frac{e}{d} + \left(\frac{L}{d}\right)^2 \left\{\left[\frac{1}{8} - \frac{1}{\pi^2}\left(\frac{16}{9\sqrt{3}} + \frac{1}{9}\right)\right](\epsilon_1 - \epsilon_2)_{L/2} + \frac{16}{9\sqrt{3}\,\pi^2}(\epsilon_1 - \epsilon_2)_{L/6} + \frac{1}{9\,\pi^2}(\epsilon_1 - \epsilon_2)_o\right\} \tag{10c}$$

Each of the above equations represents a straight line on the chart of $(\epsilon_1 - \epsilon_2)$ versus $(e + y)/d$. For a given value of axial stress on the column, equation (10a) establishes the difference in extreme fiber strains at the end of the column, which is used in the equation (10b) to find the difference in extreme fiber strains at the one-third point. Both of these values are then substituted in the equation (10c), which determines the deflection and strain at the center of the column.

In order to simplify the task of computing the relationship between load, moment arm, and curvature from the stress-strain diagram, it was assumed that no stress reversal occurred at a point after the stress at that point had entered the plastic stress range. The strain measurements indicated this assumption to be correct for all but the shortest columns tested with the smallest eccentricities. One consequence of this assumption is that the limiting load for zero eccentricity in the plastic stress range is calculated to be the tangent modulus critical load rather than the somewhat higher value which results if strain reversal is taken into account.[5] An additional assumption used to simplify the calculations for the rectangular tube was that all the resistance to bending was concentrated at the centroids of the flanges.

TABLE I

MECHANICAL PROPERTIES
Eccentrically-Loaded 61S-T6 Alloy Columns

Material	Tensile Strength ksi	Tensile Yield Strength (set - 0.2%)	Elongation in 2 in., per cent	Compressive Yield Strength (set - 0.2%) ksi
Rolled Rectangular Bar	44.0[a]	40.6[a]	17[a]	38.9[c]
Drawn Rectangular Tube	46.5[b]	40.8[b]	16[b]	40.8[d]

a Determined from standard 1/2-in. diameter round tension specimens, see ASTM Standards, 1952, Part II, E8-52T, Fig. 7, p. 1211.

b Determined from standard 1/2-in. wide rectangular tension specimens, cut from tube wall and machined to uniform thickness, ibid., Fig. 6, p. 1211.

c Determined from 1/2-diameter round compressive specimens, 1.875 in. long.

d Determined from 5/8-in. wide 2.630 in. long rectangular compressive specimens, cut from tube wall and machined to uniform thickness (tested in subpress with Montgomery-Templin type jig, ibid., E9-52T, Fig. 3, p. 1224, and Fig. 4a, p. 1226)

TABLE II

SUMMARY OF TEST RESULTS
Eccentrically Loaded 61S-T6 Alloy Columns

RECTANGULAR BAR SPECIMENS

Specimen No.	Length, in.	Eccentricity, in.	Maximum Load, kips	Calculated Maximum Load,[d] kips	Variation of Calculated from Measured Max. Load, per cent
B16[a]	15.1	0.002	94.4	91.7	-2.9
B15[a]	15.1	0.045	78.5	77.7	-1.0
B17[a]	15.1	0.160	63.5	60.0	-5.5
B20[a]	15.1	0.245	55.0	52.7	-4.2
B19[a]	15.1	0.330	48.2	47.4	-1.7
B18[a]	14.9	0.425	43.0	42.5	-1.2
B11[a]	25.1	0.002	75.2	71.1	-5.5
B13[a]	25.1	0.055	55.8	53.7	-3.8
B12[a]	25.1	0.155	43.8	42.6	-2.7
B14[a]	25.1	0.450	30.5	29.3	-3.9
B5[b]	37.6	0.005	35.9	34.5	-3.9
B8[b]	37.6	0.045	31.3	31.6	+1.0
B7[b]	37.6	0.155	26.0	26.1	+0.4
B6[b]	37.6	0.350	22.1	21.7	-1.8
B9[b]	37.6	0.445	19.8	19.5	-1.5
B1[b]	47.6	0.000	23.2	22.0	-5.2
B4[b]	47.6	0.035	21.1	21.1	0
B2[b]	47.6	0.175	17.8	17.6	-1.1
B3[b]	47.6	0.310	16.2	16.1	-0.6
B4[c]	45.1	0.645	14.6	14.3	-2.1
				avg.	-2.4

RECTANGULAR TUBE SPECIMENS

Specimen No.	Length, in.	Eccentricity, in.	Maximum Load, kips	Calculated maximum Load,[d] kips	Variation of Calculated from Measured Max. Load, per cent
T16[a]	20.0	0.002	81.5	80.5	-1.2
T14[a]	20.0	0.100	70.5	67.8	-3.8
T15[a]	20.0	0.300	56.6	53.0	-6.4
T12[a]	35.0	0.003	74.2	70.9	-4.4
T9[a]	35.0	0.095	56.6	54.9	-3.0
T10[a]	35.0	0.305	43.8	41.5	-5.3
T11[a]	35.0	0.620	33.5	31.9	-4.8
T8[b]	52.5	0.002	46.8	45.4	-3.0
T5[b]	52.5	0.095	37.7	36.3	-3.7
T6[b]	52.5	0.290	30.3	29.4	-3.0
T7[b]	52.5	0.545	25.1	24.5	-2.4
T4[b]	67.5	0.007	27.6	27.7	+0.4
T2[b]	67.5	0.095	25.2	24.9	-1.2
T3[b]	67.5	0.280	22.0	21.4	-2.7
T1[b]	67.4	0.570	18.8	18.2	-3.2
T2[c]	65.0	0.915	17.4	16.6	-4.6
				avg.	-3.3

a Tested on spherically-seated, hydraulically-supported platens, with centers of rotation at ends of column.

b Tested on knife-edge platens with the centers of rotation 1.25 in. beyond the ends of the column. Lengths listed include this extra 2 1/2 inches.

c Tested with knife edges machined on the ends of the column.

d Based on equation (2), Fig. 8.

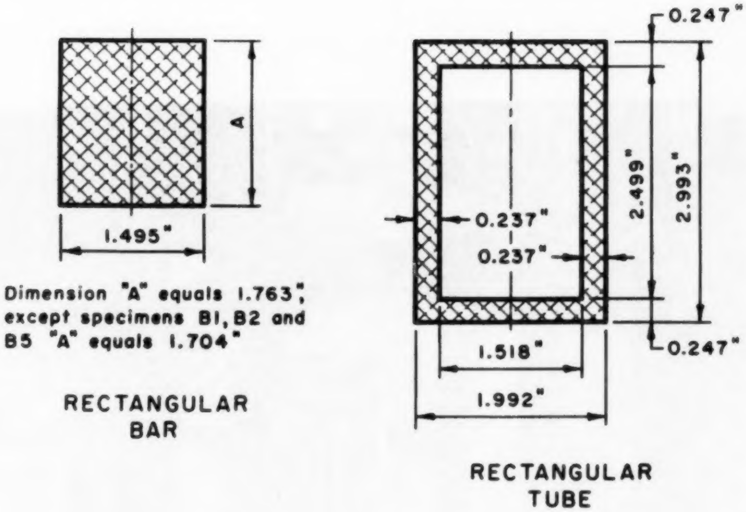


FIG. 1 - AVERAGE CROSS SECTION DIMENSIONS OF COLUMN SPECIMENS

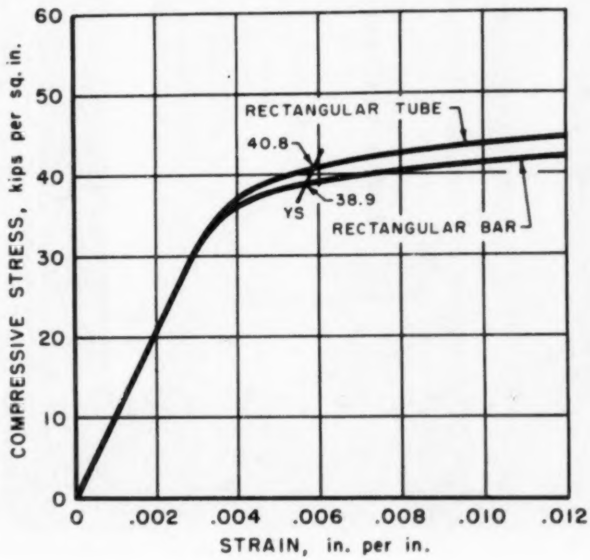


FIG. 2 - COMPRESSIVE STRESS-STRAIN CURVES FOR 61S-T6 ALUMINUM ALLOY COLUMN SPECIMENS

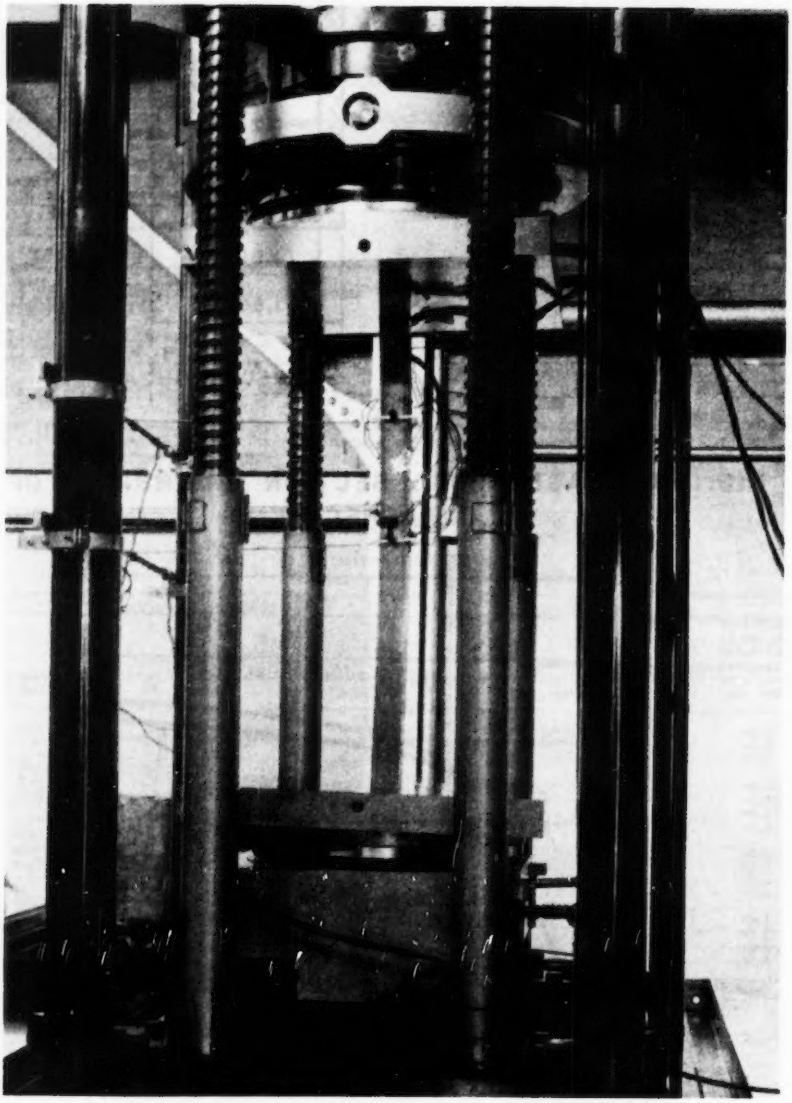


Fig. 3. Rectangular Tube Column Specimen T10 After Loading to Failure on Hydraulically Supported, Spherically Seated Platens.
Length = 35 in., Eccentricity = 0.305 in.

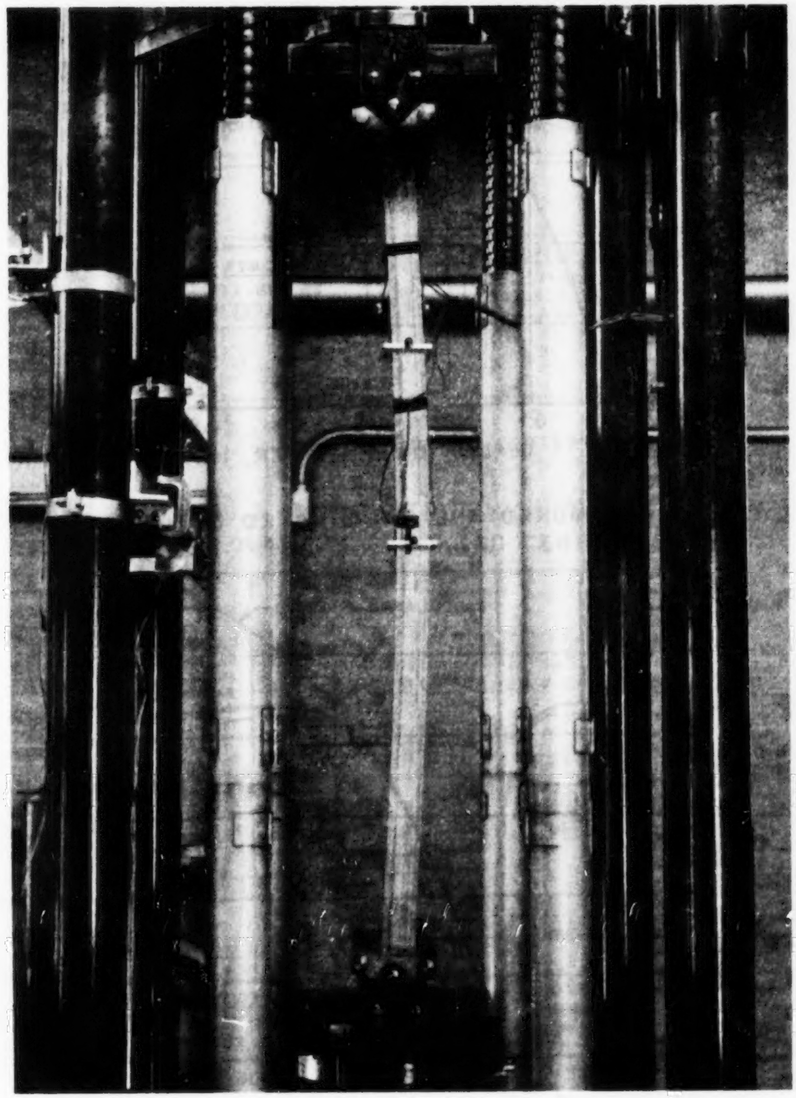
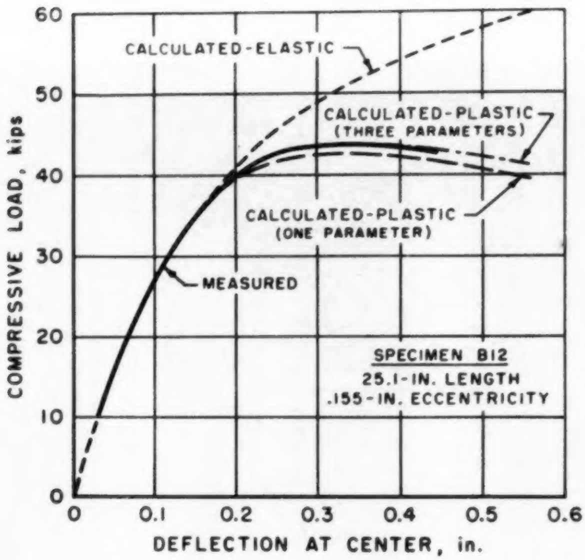


Fig. 4. Rectangular Tube Column Specimen T6 After Loading to Failure on Knife-Edge Platens.
Length = 52.5 in., Eccentricity = 0.290 in.



5 - MEASURED AND CALCULATED DEFLECTIONS

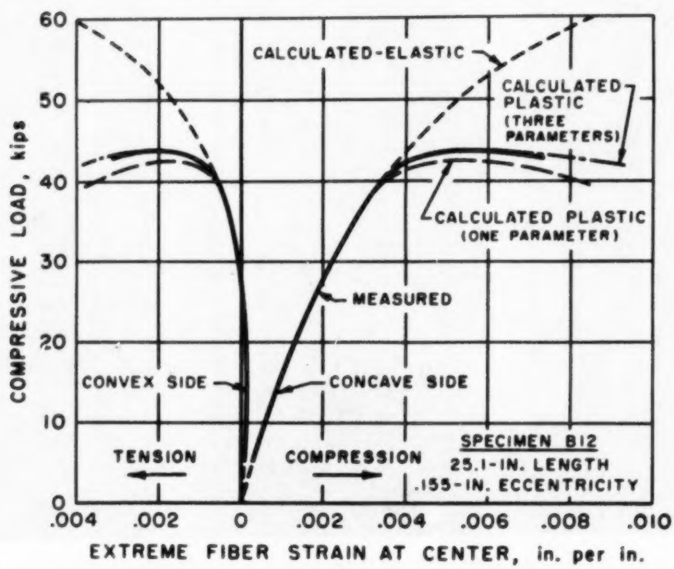


FIG. 6 - MEASURED AND CALCULATED STRAINS

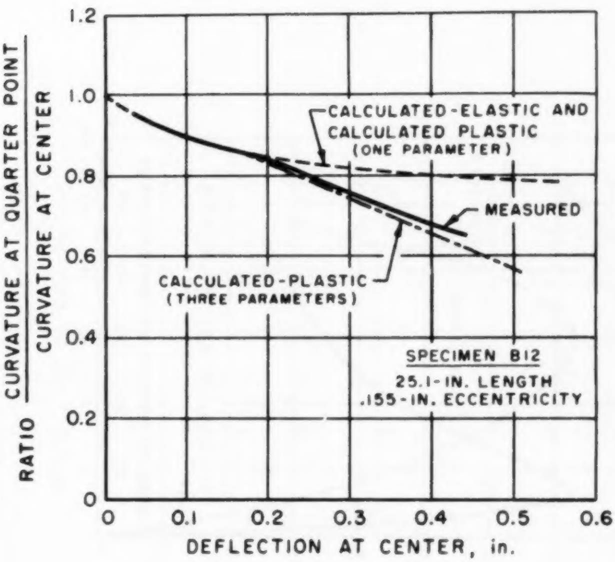
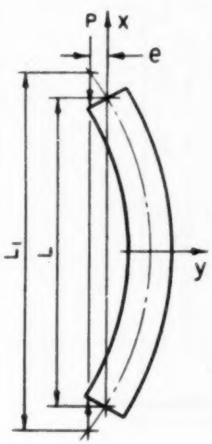


FIG. 7 - RATIO OF CURVATURES AT QUARTER POINT AND CENTER



MATHEMATICAL EXPRESSIONS ASSUMED TO REPRESENT DEFLECTED SHAPE OF COLUMN

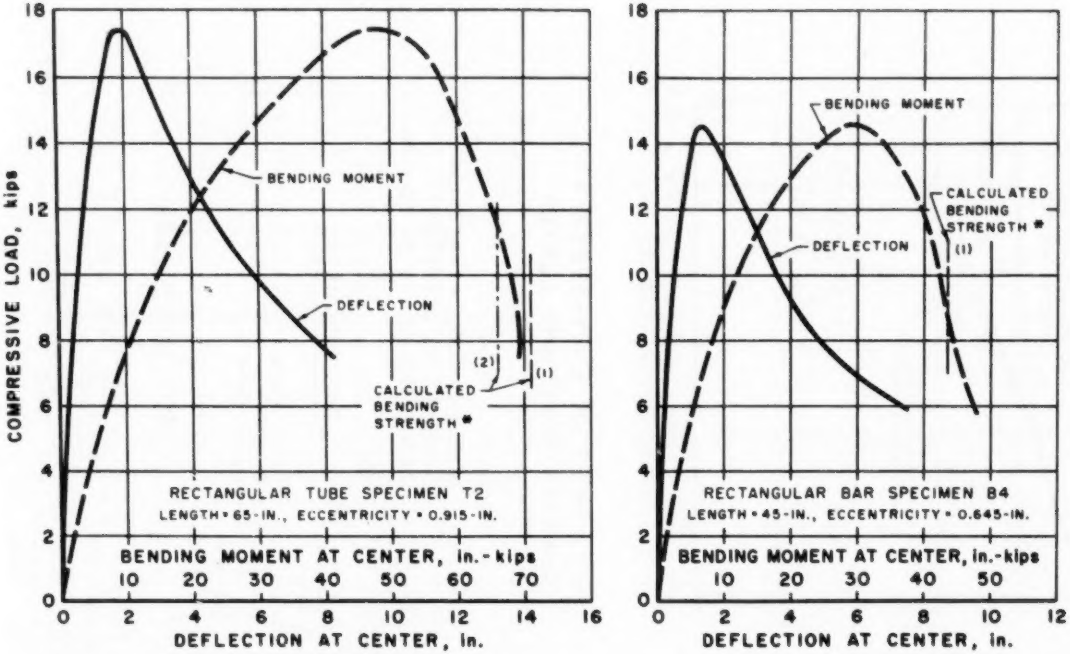
ONE PARAMETER

$$y = a \cos \frac{\pi x}{L_1} - e \qquad (2)$$

THREE PARAMETERS

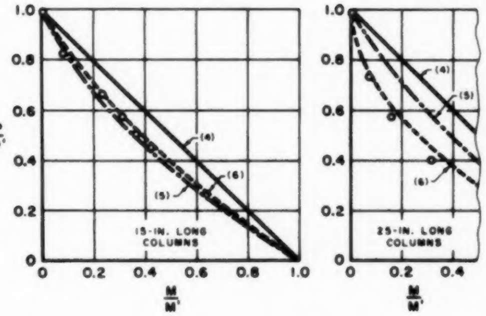
$$y = a_0\left(1 - \frac{4x^2}{L^2}\right) + a_1 \cos \frac{\pi x}{L} + a_2 \cos \frac{3\pi x}{L} \qquad (3)$$

FIG. 8 - SHAPE OF ECCENTRICALLY LOADED COLUMN IN DEFLECTED POSITION



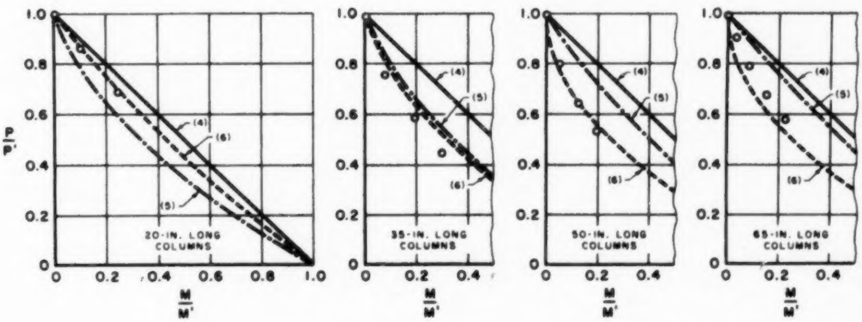
* CALCULATED ULTIMATE BENDING STRENGTH (1) BASED ON THE ASSUMPTION OF A RECTANGULAR DISTRIBUTION OF STRESS EQUAL TO THE TENSILE STRENGTH, (2) ON THE ASSUMPTION THAT THE BENDING STRESS (Mc/I) REACHES THE TENSILE STRENGTH AT THE CENTROID OF THE TENSION FLANGE

FIG. 9 - DEFLECTION AND BENDING MOMENT IN COLUMN SPECIMENS TESTED WITH KNIFE-EDGE ENDS



INTERACTION FORMULAS: (4) $\frac{P}{P'} + \frac{M}{M'} = 1$ (5) $\frac{P}{P'} + \frac{M}{M'(1-PS/M'A)} = 1$ (6) $\frac{P}{P'} + \frac{M}{M'(1-P/P_E)} = 1$

FIG.10 – INTERACTION DIAGRAMS FOR RECTANGULAR BARS TESTED AS ECCENTRICALLY LOADED COLUMNS



INTERACTION FORMULAS: (4) $\frac{P}{P'} + \frac{M}{M'} = 1$ (5) $\frac{P}{P'} + \frac{M}{M'(1-PS/M'A)} = 1$ (6) $\frac{P}{P'} + \frac{M}{M'(1-P/P_E)} = 1$

FIG.11 – INTERACTION DIAGRAMS FOR RECTANGULAR TUBES TESTED AS ECCENTRICALLY LOADED COLUMNS